EFFECTIVE NON-VANISHING OF ASYMPTOTIC ADJOINT SYZYGIES

XIN ZHOU

Introduction

The purpose of this paper is to establish an effective non-vanishing theorem for the syzygies of an adjoint-type line bundle on a smooth variety, as the positivity of the embedding increases. In particular, we give an answer to Problem 7.9 in [E-L] in this setting.

Let X be a smooth projective variety of dimension n over $\mathbb{C}$, and let L be a very ample line bundle on X. Then L defines an embedding:

$$X \hookrightarrow \mathbb{P}^{r(L)} = \mathbb{P}H^0(X, L) = \text{Proj } S$$

where $r(L) = h^0(X, L) - 1$ and $S = \text{Sym}H^0(X, L)$. Given a divisor B on X, write:

$$R(X, B; L) = \bigoplus_m H^0(X, mL + B)$$

which is viewed as a finitely generated graded S-module. We will be interested in the syzygies of $R(X, B; L)$ over S. Specifically, R has a graded minimal free resolution

$$\mathbb{F} : ... \to F_p \to ... \to F_0 \to R \to 0$$

where $F_p = \oplus_j S(-a_{p,j})$ is a free S-module. Write $K_{p,q}(X, B; L)$ for the finite dimensional vector space of minimal p-th syzygies of degree $(p + q)$, so that:

$$F_p \cong \bigoplus_q K_{p,q}(X, B; L) \otimes_{\mathbb{C}} S(-p - q)$$

Ein and Lazarsfeld [E-L] studied these groups for $L = L_d = dA$ when A is a very ample line bundle and d is very large. It is elementary that:

- For $q > n + 1$, $K_{p,q}(X, B; L_d) = 0$
- For $q = 0$ or $q = n + 1$, $K_{p,q}(X, B; L_d) \neq 0$ for only finitely many values of p, which can be determined completely.

So the interesting question is when $K_{p,q}(X, B; L_d)$ is nonzero for $1 \leq q \leq n$. The first main result of [E-L] was that if one fixes $1 \leq q \leq n$, and if $d \gg 0$, then

$$K_{p,q}(X, B; L_d) \neq 0$$

for

$$O(d^{q-1}) \ \leq \ p \ \leq \ r(d) - O(d^{n-1}).$$

Remark 0.1. Another result of similar flavor on surfaces is [E-G-H-P, Prop. 3.4].

The second main result of [E-L] was the effective statement for $X = \mathbb{P}^n$, $B \in |\mathcal{O}_{\mathbb{P}^n}(k)|$, $1 \leq q \leq n$ and d large. Specifically,

$$K_{p,q}(\mathbb{P}^n, B; L_d) \neq 0$$

for

$$\binom{q+d}{d} - \binom{d-k-1}{q} - q \leq p \leq \binom{d+n}{n} - \binom{d+n-q}{n-q} + \binom{k+n}{n-q} - q - 1$$

Our purpose here is to show that for an adjoint type divisor $B = K_X + bA$ with $b \geq n+1$, one can in fact obtain an effective statement for arbitrary X which specializes to the statement above on Veronese syzygies. Before stating the theorem, we fix some notations. Given very ample line bundles $H_1, ..., H_c$ on X, let Z be a complete intersection of divisors from $|H_1|, ..., |H_c|$. Write:

$$\phi(H_1, ..., H_c; L_d) = h^0(Z, L_d).$$

Via the Koszul resolution of $\mathcal{O}_Z$, for sufficiently large d, $\phi(H_1, ..., H_c; L_d)$ can be expressed, independent of the choice of the particular divisors, which will be the case for us, as an alternating sum of terms of the form $h^0(X, dL - \Sigma_{j \in J} H_j)$, where $J \subseteq \{1, ..., c\}$. The following two special cases appear in the statement of our main result:

$$n_d := \phi(-K_X - (n-q)A + B, \underbrace{A, ..., A}_{n-q}; L_d)$$

$$N_d := \phi((d-q)A - B, \underbrace{A, ..., A}_{q}; L_d)$$

Our main result is:

Theorem. *Fix $1 \leq q \leq n$. Then for sufficiently large d,*

$$K_{p,q}(X, B; L_d) \neq 0 \tag{0.1}$$

for every value of p satisfying:

$$n_d - q \leq \ p \ \leq h^0(X, L_d) - N_d - q - 1 \tag{0.2}$$

It may be instructive to see an example of how the theorem works. Let $X = \mathbb{P}^2$, and put $B = 0$, $A = \mathcal{O}_X(1)$. Then we're looking at the minimal free resolution of the image of $\mathbb{P}^2$ in its d-th Veronese embedding, and we work out the statement in the case $q = 1$.

- In the lower bound, Z is 2 points and $n_d = h^0(Z, \mathcal{O}_{\mathbb{P}^2}(d)) = 2$.
- In the upper bound, Z consists of d points and $N_d = h^0(Z, \mathcal{O}_{\mathbb{P}^2}(d)) = d$.

So for d-th Veronese embedding of $\mathbb{P}^2$ with large d, the theorem asserts that:

$$K_{p,1}(X, 0; dL) \neq 0, \text{for } \ 1 \leq p \leq \binom{d+2}{2} - d - 2$$

which was a result of Ottaviani and Paoletti, cf. [O-P]. More generally, one can recover [E-L, Thm. 6.1]. (See Example 2.1 below.)

The proof of the theorem follows the line of attack of [E-L], which involves constructing secant varieties that exhibit syzygies. The main new observation here is that for adjoint B, one can work with secant varieties that do not vary with d. This greatly simplifies the calculations, and gives an effective statement which specializes to the case of Veronese embeddings. For a number of facts we use in the proof, we will refer to [E-L] when appropriate, instead of repeating the arguments in detail. In §1, we give a proof of the main theorem. In §2, we work out some examples.

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1. Proof of the main result

We first give an alternative definition of $K_{p,q}(X,B;L)$ used in the proof. Throughout, X is a smooth complex projective variety of dimension n and L and A are very ample divisors on X. (We will be setting $L = dA$ later.) Let

$$V = H^0(X,L)$$

and write V_X for $V \otimes_{\mathbb{C}} \mathcal{O}_X$, the trivial vector bundle on X modeled on V.

Fix an integer $b \geq n+1$ and set

$$B = K_X + bA + P, \tag{1.1}$$

where K_X is a canonical divisor on X and P is trivial or globally generated (we have to include this extra case for the application of duality after Prop. 1.12). Note that the higher cohomologies of B vanish thanks to Kodaira vanishing. As in [G] and [E-L], define $K_{p,q}(X,B;L)$ to be the cohomology at the middle, of the following complex:

$$\wedge^{p+1}V \otimes H^0(B+(q-1)L) \to \wedge^p V \otimes H^0(B+qL) \to \wedge^{p-1}V \otimes H^0(B+(q+1)L)$$

As motivated in the introduction, we will fix the index $q \in [1,n]$. As is well-known, these Koszul cohomology groups are governed by the cohomologies of the vector bundle M_L on X defined by the exact sequence:

$$0 \to M_L \to V_X \to L \to 0 \tag{1.2}$$

Proposition 1.1. *For* $1 \leq q \leq n$, $K_{p,q}(X,B;L) = H^q(\wedge^{p+q} M_L \otimes \mathcal{O}_X(B))$.

Proof. Recalling that B has vanishing higher cohomologies, the conclusion follows as in [E-L, Prop. 3.2, Prop. 3.3]. See also [E-L 93, Sect 1], [E, Thm. 5.8]. □

Next, we recall the following construction from [E-L, §3]. Suppose a quotient $\pi : V \twoheadrightarrow W$ of V with $\dim W = w$ and a subscheme Z of X satisfy:

$$Z = X \cap \mathbb{P}(W)$$

in $\mathbb{P}(V)$ scheme-theoretically. Then we have the diagram:

$$\begin{tikzcd}
0 \arrow[r] & M_L \arrow[r] \arrow[d] & V \otimes O_X \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma_W \arrow[r] & W \otimes O_X \arrow[r] & L \otimes O_Z \arrow[r] & 0
\end{tikzcd} \tag{1.3}$$

whose bottom exact sequence defines Σ_W, a torsion-free sheaf on X of rank w. Furthermore, as in [E-L, §3], $\wedge^w \Sigma_W$ maps onto $\mathcal{I}_{Z/X}$ and one gets a surjective map:

$$\sigma_\pi : \wedge^w M_L \to \mathcal{I}_{Z/X} \tag{1.4}$$

We modify [E-L, Def. 3.8]:

Definition 1.2. We say that W carries weight q syzygies for B if the map induced by σ_π:

$$H^q(X, \wedge^w M_L(B)) \to H^q(X, \mathcal{I}_{Z/X}(B))$$

is surjective. (We also say the same for $q = 0$ for notational convenience even though it isn't necessarily directly related to syzygies.)

Remark 1.3. If for some $q \geq 1$, $H^q(X, \mathcal{I}_{Z/X}(B)) \neq 0$ and W carries weight q syzygies for B, then combining Prop. 1.1 and Def. 1.2 gives us $K_{w-q,q}(X,B;L) \neq 0$

The lemma below describes the same kind of inductive behavior as in [E-L, Thm. 3.10], with our new definition. Let's recall some notations first. The assumption under which we'll apply the lemma is that X has dimension at least 2. (The dimension one case is completely understood, cf. [E-L, Prop. 5.1].) Take a general divisor $\overline{X} \in |A|$ so that $\overline{X}$ is smooth, irreducible and so that (1.3) remains exact after tensoring with $O_{\overline{X}}$. Let

$$V' = V \cap H^0(X, I_{\overline{X}/X}(A))$$

with the intersection inside V. Set $W' = \pi(V')$. Write

$$\overline{V} = V/V',\ \overline{W} = W/W',\ \overline{L} = L|_{\overline{X}},\ \overline{B} = B|_{\overline{X}},\ \overline{Z} = Z \cap X$$

As in [E-L, (3.14)], we get the analogue of (1.3) above for the barred objects and we have the surjection:

$$\overline{\sigma} : \wedge^{\overline{w}} M_{\overline{V}} \to I_{\overline{Z}/\overline{X}}$$

so we can study the behavior of $\overline{W}$ with respect to carrying syzygies. In fact, the same kind of argument as in the proof of [E-L, Thm. 3.10] yields:

Lemma 1.4. *For $q \geq 1$, if $\overline{W}$ carries weight $q-1$ syzygies for $\overline{B} + \overline{A}$ on $\overline{X}$ and if*

$$H^q(X, I_{Z/X}(B + A)) = 0,$$

then W carries weight q syzygies for B on X.

We next give an analogue of [E-L, Def. 5.3]. Let $c = n + 1 - q$. Take divisors

$$D_1 \in | - K_X - (c-1)A + B|,\ D_2 \dots D_c \in |A|,$$

and let $Z = D_1 \cap D_2 ... \cap D_c$.

Definition 1.5. We say that Z is adapted to (X, B, L, n, q), if Z is constructed from X, B, A, n, q as above and the $\{D_i\}$ intersect transversely.

To get started, we take D_i general in its linear series. Then we have:

Proposition 1.6.
(i) One has, for every $J \subsetneq \{1, 2, ...c\}$ and $i > 0$, $H^i(X, -\Sigma_{j \in J} D_j + B) = 0$.
(ii) $H^q(X, I_{Z/X}(B)) \neq 0$ and $H^q(X, I_{Z/X}(B + A)) = 0$.
(iii) If $\overline{Z} = Z \cap \overline{X}$, then $\overline{Z}$ is adapted to $(\overline{X}, \overline{B} + \overline{A}, \overline{A}, n-1, q-1)$.

Proof. (i) Thanks to the choice of D_i, this follows from Kodaira vanishing.
(ii) Since X is a smooth projective variety and the D_i meet transversely, Z is a complete intersection. So $I_{Z/X}$ is resolved by the Koszul complex with j-th term $\wedge^j E$ where $E = \oplus_{i=1}^c O_X(-D_i)$. Use the Koszul resolution twisted by B and (i) to find:

$$H^q(X, I_{Z/X}(B)) = H^{q+c-1}(X, B - (\Sigma_{i=1}^c D_i)) = H^n(X, K_X) \neq 0$$

The other claim follows similarly using Kodaira vanishing.
(iii) Using adjunction, we have:

$$\overline{B} = K_{\overline{X}} + (b-1)\overline{A} + \overline{P}$$

so $\overline{B}$ has the shape as in (1.1). As $\overline{X}$ is general, and D_i's are general, we can assume $\{\overline{D_i}\}$ meet transversely. (Similarly, in the finite number of steps in the induction, we are free to assume the corresponding divisors intersect transversely.) $\overline{D_1}$ is in the correct linear series by adjunction. The rest is immediate.

$\square$

Now let $L_d = dA$ and we take d large enough so that for any i, $H^i(X, L_d - iA) = 0$.

Definition 1.7. We say that d satisfies the effective conditions for B, if $L_d - nA - B$ is ample.

Remark 1.8. Assume that d satisfies the effective conditions for B. Note that $\Sigma_{i=1}^c D_i = B - K_X$ by construction. Via the Koszul resolution on $\{D_i\}$ twisted by L_d and using Kodaira vanishing, the following statements hold and furthermore, they hold after cutting down by hyperplanes repeatedly until we reach the base case of the inductive proof in Prop. 1.10:

(1) The map $H^0(X, L_d) \to H^0(Z, L_d)$ is surjective, equivalently, $H^1(X, I_{Z/X}(L_d)) = 0$.
(2) The map $H^0(Z, L_d) \to H^0(\overline{Z}, L_d)$ is surjective, equivalently, $H^1(Z, L_d - A) = 0$.
(3) $H^1(X, I_{Z/X}(L_d - A)) = 0$ (and equivalently, with W' chosen below, the map $V' \to W'$ is surjective.)
(4) The map $H^0(X, L_d) \to H^0(\overline{X}, L_d)$ is surjective, or equivalently $H^1(X, L_d - A) = 0$.

Remark 1.9. In [E-L], complete intersections Z_d were chosen that varied with d. The surjectivity of the above four maps cannot be guaranteed. This resulted in the ineffectivity and a number of complications which we are able to circumvent as in the proof of Prop. 1.10.

Proposition 1.10. *For $1 \le q \le n-1$, if d satisfies the effective conditions for B, then $H^0(Z, dL)$ carries weight q syzygies for B.*

Proof. Set $W = H^0(Z, L_d)$. When d satisfies the effective conditions for B, $L_d - D_i$ is ample. So $I_{Z/X} \otimes \mathcal{O}_X(L_d)$ is globally generated. Then $Z = X \cap \mathbb{P}(W)$. Furthermore, by Rmk. 1.8, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & V' = H^0(X, L_d - A) \arrow[r] \arrow[d] & V = H^0(X, L_d) \arrow[r] \arrow[d] & \overline{V} = H^0(\overline{X}, L_d) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W' \arrow[r] \arrow[d] & W = H^0(Z, L_d) \arrow[r] \arrow[d] & \overline{W} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{1.5}$$

where

$$W' = H^0(Z, L_d - A),\ \overline{W} = H^0(\overline{Z}, L_d). \tag{1.6}$$

Moreover, when we cut down by hyperplanes as in Lemma 1.4, we obtain the corresponding diagrams in lower dimensions.

We prove the proposition by induction on q. Notice that by construction, Z is always of dimension $q-1$. Our assumption $\dim X \ge 2$ (below Remark 1.3) will always be satisifed, because $\dim X = n \ge q+1 \ge 2$. When $q = 1, Z$ consists of points. $\overline{Z} = \phi, \overline{W} = 0$. So the conclusion is trivially true for $q = 0$. Then the conclusion is true for $q = 1$ by Prop. 1.6 (ii) and Lemma 1.4. Then apply Lemma 1.4 repeatedly.

□

The previous propsotion gives $K_{p,q}(X, B; L_d) \neq 0$ for a specific p. We in fact get a range of non-vanishings by enlarging W but keeping $\overline{W}$ fixed, as in [E-L, Thm. 3.11]. The result is:

Proposition 1.11. *Fix $1 \le q \le n-1$. If d satisfies the effective conditions for B, and $h^0(X,L_d) - h^0(Z,L_d) > n$ then $K_{p,q}(X,B;L_d) \neq 0$ for p in the range:*

$$h^0(Z,L_d) - q \ \le \ p \ \le \ h^0(X,L_d - A) + h^0(\overline{Z},L_d) - q \qquad \square$$

Now we work to apply duality to the above proposition. In order to prove Thm. 1.15, we combine the above proposition with the range we get using duality.

Let $B' = L_d - B + K_X$. Notice that when d is large, B' will be of the form

$$K_X + b'A + P' \tag{1.7}$$

with $b' \ge n+1$ and P' globally generated. We work with d large enough so that B' is indeed of this form.

Proposition 1.12. *For $1 \le q \le n$,*

$$K_{p,q}(X,B;L_d) = K_{r_d-p-n,n-q}(X,B';L_d)^*$$

where $r_d = h^0(X,L_d) - 1$.

Proof. See [G, Thm. 2.c.1] for $1 \le q \le n-1$ and combine with [E-L, Rmk. 3.4] for $q = n$. $\square$

We want to apply Prop. 1.11 to B', so we study Z' adapted to $(X,B',A,n,n-q)$. Denote by D'_i, the general divisors in the corresponding linear series.

Lemma 1.13. *d satisfies the effective condition for B'.*

Proof. By definition of B' and (1.1), $L_d - nA - B' = L_d - nA - (L_d - B + K_X) = B - nA - K_X$, which is ample. $\square$

Remark 1.14. Note that when d is large, the other assumption in Prop. 1.11 is also satisfied for B'. The interested reader can check this keeping in mind that Z' is always contained in a divisor in $|A|$ and use surjections as those in Rmk. 1.8.

Finally, we arrive at the main result:

Theorem 1.15. *Fix $1 \le q \le n$. For sufficiently large d, if*

$$h^0(Z,L_d) - q \le p \le h^0(X,L_d) - h^0(Z',L_d) - q - 1$$

then $K_{p,q}(X,B;L_d) \neq 0$

Proof. For $1 \le q \le n-1$, by Prop. 1.11, we have $K_{p,q}(X,B;L_d) \neq 0$ for

$$h^0(Z,L_d) - q \le p \le h^0(X,L_d - A) + h^0(\overline{Z},L_d) - q \tag{1.8}$$

By Lemma 1.13 and Remark 1.14, we can apply Prop. 1.11 to B' and we have nonvanishings for p in range:

$$h^0(Z',L_d) - (n-q) \le r - p - n \le h^0(X,L_d - A) + h^0(\overline{Z'},L_d) - (n-q)$$

ie.

$$r_d - n - (h^0(X,L_d - A) + h^0(\overline{Z'},L_d) - (n-q)) \le p \le r_d - n - (h^0(Z',L_d) - (n-q)) \tag{1.9}$$

Now we show that the right hand side of (1.8) is of higher order than the left hand side of (1.9), so the two ranges overlap. The right hand side of (1.8) has order $O((d-1)^n) = O(d^n)$. On the left hand side of (1.7), terms of order d^n appear in r_d and $h^0(X,L_d - A)$ with the same coefficient and hence cancel. Therefore, the order is bounded by $O(d^{n-1})$. Hence,

asymptotically, we have nonvanshing for everything between the left hand side of (1.8) and the right hand side of (1.9).

For $q = n$, by Prop. 1.12, it is equivalent to the case of $q = 0$ for B'. By [E-L, Prop. 3.3], $K_{p,n}(X, B; dL) \neq 0$ if and only if

$$0 \leq r(L_d) - p - n \leq r(B').$$

This unwinds to be

$$h^0(X, L_d) - h^0(X, L_d - bA) - n \leq p \leq h^0(X, L_d) - n - 1$$

Therefore, the statement is not only true, but also sharp for $q = n$. □

2. Examples

We conclude by working out the statement of the main theorem in some interesting special cases.

2.1. **Projective space.** Take $X = \mathbb{P}^n$. Let B be a divisor in $|\mathcal{O}_{\mathbb{P}^n}(k)|$. Assume $k \geq 0$, so that B satisfies (1.1). In this case, Z is a complete intersection of $(n-q)$ divisors in $|\mathcal{O}_{\mathbb{P}^n}(1)|$ and a divisor in $|-K_X - (c-1)L + B| = |\mathcal{O}_{\mathbb{P}^n}(k+q+1)|$. So,

$$h^0(Z, dL) = \binom{q+d}{d} - \binom{d-(k+q+1)+q}{q}$$

Z' is a complete intersection of q divisors in $|\mathcal{O}_{\mathbb{P}^n}(1)|$ and a divisor in $|-K_X-(c'-1)L+B'| = |(d-k-q)L|$. So,

$$\begin{aligned} h^0(Z', dL) &= \binom{d+n-q}{n-q} - \binom{n-q+d-(d-k-q)}{d-(d-k-q)} \\ &= \binom{d+n-q}{n-q} - \binom{k+n}{n-q} \end{aligned}$$

So by Thm 1.15, for large d, $K_{p,q}(X, B; dL) \neq 0$ for p in the range:

$$\binom{q+d}{d} - \binom{d-k-1}{q} - q \leq p \leq \binom{d+n}{n} - \binom{d+n-q}{n-q} + \binom{k+n}{n-q} - q - 1$$

This corollary of Thm. 1.15 coincides with the result [E-L, Thm. 6.1], but the proof is simpler since we are able to take Z independent of d and is a specialization of the general result.

2.2. **Product of projective spaces.** Let $X = \mathbb{P}^s \times \mathbb{P}^t$, so $n = \dim X = s + t$. Divisors B satisfying (1.1) are of type (u, v) with $u \geq t+1, v \geq s+1$. As mentioned in the introduction, we can compute n_d, and N_d in the statement of the theorem (cf. (0.2)) through the Koszul resolution. Via the Kunneth formula, we find, for $1 \leq q \leq n$, sufficiently large d:

$$K_{p,q}(X, B; L_d) \neq 0$$

for p in range:

$$\begin{aligned} &\sum_{i=0}^{s+t-q} (-1)^i \binom{s+t-q}{i} \binom{d-i+s}{s} \binom{d-i+t}{t} \\ &+ \sum_{i=0}^{s+t-q} (-1)^{i+1} \binom{s+t-q}{i} \binom{d-i-u-q+t+s}{s} \binom{d-i-v-q+s+t}{t} - q \\ &\leq\ p\ \leq \end{aligned}$$

$$\binom{d+s}{s}\binom{d+t}{t} - \left(\sum_{i=0}^{q}(-1)^i\binom{s+t-q}{i}\binom{d-i+s}{s}\binom{d-i+t}{t}\right)$$
$$-\left(\sum_{i=0}^{q}(-1)^{i+1}\binom{s+t-q}{i}\binom{q+u-i+s}{s}\binom{q+v-i+t}{t}\right) - q - 1$$

Remark 2.1. The minimal free resolutions of classical Segre or multi-Segre embeddings with line bundles of type $(1, ..., 1)$ are much studied. Rubei studies $\mathbb{P}^1 \times ... \times \mathbb{P}^1$ in [R] with respect to the behavior N_p. Netay studies $\mathbb{P}^m \times \mathbb{P}^n$ in [N], giving an algorithm for computing the groups as representations. Snowden studies $\mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_m}$ in [S] and proves a finiteness theorem as we vary the number of direct product factors and the dimensions of the projective spaces.

2.3. **Grassmannian Gr(2,4).** Let $X = \mathrm{Gr}(2,4)$, 2 dimensional subspaces of $\mathbb{C}^4$. Then $\mathrm{Pic}(X) = \mathbb{Z}$ and $\mathcal{O}_X(K_X) = \mathcal{O}_X(-4)$. For the Plucker embedding, the embedding line bundle $A = O_X(1)$. Assume B is of type $\mathcal{O}_X(k)$ where $k \geq 1$ (satisfying (1.1)). Then:

$$h^0(Z, L_d) = \sum_{i=0}^{4-q}(-1)^i\binom{4-q}{i}h^0(X, \mathcal{O}_X(d-i)) + \sum_{i=0}^{4-q}(-1)^{i+1}\binom{4-q}{i}h^0(X, \mathcal{O}_X(d-i-(k+q)))$$

$$h^0(Z', L_d) = \sum_{i=0}^{q}(-1)^i\binom{q}{i}h^0(X, \mathcal{O}_X(d-i)) + \sum_{i=0}^{q}(-1)^{i+1}\binom{q}{i}h^0(X, \mathcal{O}_X(k+q-i))$$

Using the Borel-Weil-Bott theorem, $H^0(\mathcal{O}_X(m))$ corresponds to GL_4-representations corresponding to rectangular Young diagrams with 2 rows and m columns. The dimension of these representations are given by:

$$f(m) := \frac{(m+1)(m+2)(m+2)(m+3)}{12}$$

(cf. [F-H, Thm 6.3 (1)]). Combine with the expressions for $h^0(Z, L_d)$ and $h^0(Z', L_d)$, we get for $1 \leq q \leq 4$ and sufficiently large d, $K_{p,q}(X, B; L_d) \neq 0$ for p in range:

$$\sum_{i=0}^{4-q}(-1)^i\binom{4-q}{i}f(d-i) + \sum_{i=0}^{4-q}(-1)^{i+1}\binom{4-q}{i}f(d-i-k-q) - q$$
$$\leq\ p \leq$$
$$r(d) - \left(\sum_{i=0}^{q}(-1)^i\binom{q}{i}f(d-i)\right) - \left(\sum_{i=0}^{q}(-1)^{i+1}\binom{q}{i}f(k+q-i)\right) - q - 1$$

The same can be done in principle for any Grassmannian as requested in [E-L, Problem 7.9].

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University of Michigan, Department of Mathematics, 530 Church St., East Hall, Ann Arbor, MI 48109-1043, USA

E-mail address: paulxz@umich.edu